Geometry: Foundations and Finite Geometries

MATH 3120, Spring 2016

1 Axiomatic Systems

Consider a proof of a theorem in mathematics, say, the Pythagorean Theorem. Mathematicians cannot prove this theorem is true everywhere, all the time. Mathematicians can only operate within a given set of ground rules. Primitive ideas must be assumed to be true, not proven. More advanced theorems are built upon the basics, but different sets of primitive ideas are possible.

Formal axiomatics take a subject and analyze the best set of primitive ideas and convert them into undefined terms, defined terms and axioms. Upon this foundation, every single result, theorem and lemma must rely. Each must be proven in turn starting from the most basic to the most advanced. Geometry is typically studied as an axiomatic system, but we must be very careful how we choose. Tiny changes in one axiom can completely alter the theorems that can be proven.

1.1 The Axiomatic Method

An axiomatic system must contain a set of technical terms that are deliberately chosen as **undefined terms** and are subject to the interpretation of the reader.

All other technical terms of the system are **defined terms** whose definition rely upon the undefined terms.

The **axioms** of the system are a set of statements dealing with the defined and undefined terms and will remain unproven. The word "postulate" is a synonym and is often used in place of the word axiom. Euclid, for example, used postulates rather than axioms.

All other statements must be logical consequences of the axioms. These derived statements are called the **theorems** of the axiomatic system.

Axiomatic systems have properties the most essential of which is **consistency**. An axiomatic system is consistent provided any theorem which can be proven using the system can never logically contradict any of the axioms or previously proved theorems.

An idividual axiom is **independent** if it cannot be proven by use of the other axioms. An axiomatric system is independent provided each of its axioms is independent. For nearly $2,000$ years, mathematicians wondered if Euclid's Parallel Postulate was actually a theorem we could prove using his other postulates. It's not, but the search for the proof of it's dependence or independence led to the discovery of many important theorems in Euclidean geometry and, eventually, to the creation of non-Euclidean geometries.

Mathematicians generally prefer independent sets of axioms. However, these systems can be tedious to work with because each result (which the layman considers to be trivially true) must be proven individually. This labor-intensive process can be shortened by taking a few essential theorems as axioms. Though the system will be dependent, students of mathematics will be able to advance quickly to the proof of non-trivial results.

An axiom set must also be **complete**. We must have enough axioms that every theorem that can possibly be stated using the terms and axioms of the system can be proven either true or false. Proving the completeness of an axiomatic system is quite difficult and beyond the scope of this course. However, you can see the importance of this idea. We would like for our notions of geometry to all be proveable, so our set of axioms must be robust enough to establish results that match the physcial properties we need for 2D and 3D geometry. Physics relies upon geometry. Einstein said his theory of relativity would have not have possible were he not familiar with hyperbolic geometry.

1.2 Axiom Set A

Given the three undefined terms of x, y and "on":

Axiom A.1 There exist exactly five x's.

Axiom A.2 Any two distinct x's have exactly one y on both of them.

Axiom A.3 Each y is on exactly two x's.

Prove the following theorems using Axiom Set A. (Hint: think of x's as vertices of a graph and y's as edges of the graph. Draw a picture - a concrete model - to help visualize what's going on.)

Theorem TA.1 Any two y's have at most one x on both.

Theorem TA.2 Not all x's are on the same y.

Theorem TA.3 There exist exactly four y's on each x.

Theorem TA.4 For any y_1 and any x_1 not on that y_1, there exists exactly two other distinct y's on x_1 that do not contain any of the x's on y_1.

1.3 Axiom Set N

Consider an infinite set of undefined elements S and the undefined relation R.

Axiom N.1 If $a, b \in S$ and aRb, then $a \neq b$.

Axiom N.2 Given $a, b, c \in S$ such that aRb and bRc, then aRc.

The following theorems can be proven using axioms R.1 and R.2:

Theorem TN.1 Show the following interpretation is a model for the system: Let S be the set of integers and R be the relation $<$.

Theorem TN.2 Prove or disprove. The following interpretation also a model for the system: Let S be the set of integers and R be the relation $>$.

Theorem TN.3 Prove the models in TN.1 and TN.2 are isophomorphic, e.g. that there exists an isomorphism f such that, if $a < b$, then $f(a) > f(b)$.

Theorem TN.4 Prove or disprove. The following interpretation also a model for the system: Let S be the set of reall numbers and R be the relation $<$.

Theorem TN.5 Prove or disprove. The model in TN.4 is isomorphic to the model in TN.1.

1.4 Axiom Set E

Consider an infinite set of undefined elements S and the undefined relation R.

Axiom E.1 If $a \in S$ then aRa.

Axiom E.2 If $a, b \in S$ and aRb then bRa.

Axiom E.3 For every $a, b, c \in S$, if aRb and bRc, then aRc.

The following exercises and theorems refer to axioms E.1, E.2 and E.3:

Exercise 1 Devise a model for the system.

Theorem TE.1 Prove or disprove. For every $a, b, c, d \in S$, if aRb, bRc and cRd, then aRd.

Theorem TE.2 For every $a, b, c \in S$, if aRb and cRb, then aRc.

Exercise 2 Axioms 1, 2 and 3 have names. What are they? (Hint: think Abstract Algebra.)

2 Finite Geometries

We begin with some very simple axiomatic systems called finite geometries. A finite geometry has a finite set of points and lines and can be either 2D or 3D.

2.1 Four Point Geometry.

Consider an aximatic system with the undefined terms "point," "line" and "on."

Axiom 4P.1 There exist exactly four points.

Axiom 4P.2 Any two distinct points have exactly one line on both of them.

Axiom 4P.3 Each line is on exactly two points.

We will also need some **defined terms**. Defined terms should include only words whose meaning was established previously or are undefined.

Definitions for the Four Point Geometry

- **Intersect.** Two lines on the same point are said to **intersect** and are called **intersecting lines**.
- **Parallel.** Two lines that do not intersect are called **parallel** lines.
- **Concurrent.** Three or more lines that intersect on the same point are called **concurrent**.

We would normally say two lines intersect if they "contain" the same point. However, we haven't defined "contain." We use the undefined term "on" to refer both to a line being "on" some points and to points being "on" a line.

Exercise 3 Devise a model for the system.

Theorem 4P.1 In the Four Point Geometry, if two lines intersect, they have exactly one point in common.

Theorem 4P.2 The Four Point Geometry has exactly six lines.

Theorem 4P.3 In the Four Point Geometry, each point has exactly three lines on it.

Theorem 4P.4 In the Four Point Geometry, each distinct line has exactly one line parallel to it. (Hint: this is an "existence and uniqueness" proof. Generally, we start by proving existence. Then, we assume there exists a second element with the same properties, and derive a contradiction. This gives us uniqueness.)

Theorem 4P.5 Prove that there exists a set of two lines in the Four Point Geometry that contain all of the points in the geometry.

2.2 Five Point Geometry

Axioms 2 and 3 of the Five Point Geometry are identical to the Four Point Geometry. However, Axiom 1 is changed so there exist five points. Also, we'll use the same definitions as above. For clarity, here's our axiom set:

Axiom 5P.1 There exist exactly **five** points.

Axiom 5P.2 Any two distinct points have exactly one line on both of them.

Axiom 5P.3 Each line is on exactly two points.

Exercise 4 Devise a model for the Five Point Geometry.

Exercise 5 State and prove two theorems about the Five Point Geometry.

3 Fano's Geometry

Consider an axiomatic system with the undefined terms "point," "line" and "on."

Axiom F.1 There exists at least one line.

Axiom F.2 There are exactly three points on each line.

Axiom F.3 Not all points are on the same line.

Axiom F.4 There is exactly one line on any two distinct points.

Axiom F.5 There is at least point on any two distinct lines.

We will need the same definitions as before:

Definitions for Fano's Geometry

- **Intersect.** Two lines on the same point are said to **intersect** and are called **intersecting lines**.
- **Parallel.** Two lines that do not intersect are called **parallel** lines.
- **Concurrent.** Three or more lines that intersect on the same point are called **concurrent**.

Exercise 6 Devise a model for the system. Note that you can find these online with a google search. Please attempt this on your own without resorting to copying. You will learn a lot more if you spend at least half an hour working on this before resorting to copying.

Theorem F.1 In Fano's Geometry, two distinct lines have exactly one point in common. (Try to prove this without relying on the model.)

Lindsey's Lemma For every point P in Fano's Geometry, there exists a line that does not contain it.

Theorem F.2 Fano's Geometry contains exactly seven points and exactly seven lines. Help: We will break this theorem up into three lemmas. To prove these lemmas, you may assume a previous lemma is true.

> **Lemma FL.2.1** Fano's Geometry contains **at least** seven points.
>
> **Lemma FL.2.2** Fano's Geometry contains **no more than** seven points.
>
> **Lemma FL.2.3** Fano's Geometry contains **exactly** seven lines. †

Theorem F.3 Fano's Geometry contains no parallel lines.

Theorem F.4 In Fano's Geometry, each line is on exactly three points.

Theorem F.5 In Fano's Geometry, the set of all lines on a single point contains all the points in the geometry.

Theorem F.6 In Fano's Geometry, for every pair of points, there exist exactly two lines containing neither point.

Theorem F.7 In Fano's Geometry, each point is on exactly three lines. †

Theorem F.8 In Fano's Geometry, for every set of three distinct lines, there exists exactly one point not on any of the three lines.

4 Young's Geometry

Changing a single axiom can generate a very different geometry. Young's Geometry is Fano's Geometry except that Axiom 5 is different. Let's investigate what happens. Consider an axiomatic system with the undefined terms "point," "line" and "on."

Axiom Y.1 There exists at least one line.

Axiom Y.2 There are exactly three points on each line.

Axiom Y.3 Not all points are on the same line.

Axiom Y.4 There is exactly one line on any two distinct points.

Axiom Y.5 For each line l and each point P not on l, there exists exactly one line on P that does not contain any points on l.

We will use the same definitions:

Definitions for Young's Geometry

- **Intersect.** Two lines on the same point are said to **intersect** and are called **intersecting lines**.
- **Parallel.** Two lines that do not intersect are called **parallel** lines.
- **Concurrent.** Three or more lines that intersect on the same point are called **concurrent**.

Compare the new Axiom 5 to the definition of parallel. Notice that while Fano's Geometery has no parallel lines, Young's Geometry must have at least one pair of parallel lines.

Exercise 7 Devise a model for the system.

Theorem Y.1 In Young's Geometry, each point is on at least four lines.

Corollary to Y.1 or CY.1 In Young's Geometry, every point is on exactly four lines.

Theorem Y.2 In Young's Geometry, every line has exactly two distinct lines that are parallel to it. (Again, we will use lemmas to help us prove this theorem.)

Lemma YL.2.1 In Young's Geometry, if lines l and m intersect at a point P, and if line l is parallel to line n, then lines m and n share a common point Q.

Lemma YL.2.2 In Young's Geometry, two lines parallel to a third line are parallel to each other.

Theorem Y.3 Young's Geometry contains exactly twelve lines. †

Theorem Y.4 Young's Geometry contains exactly nine points. †

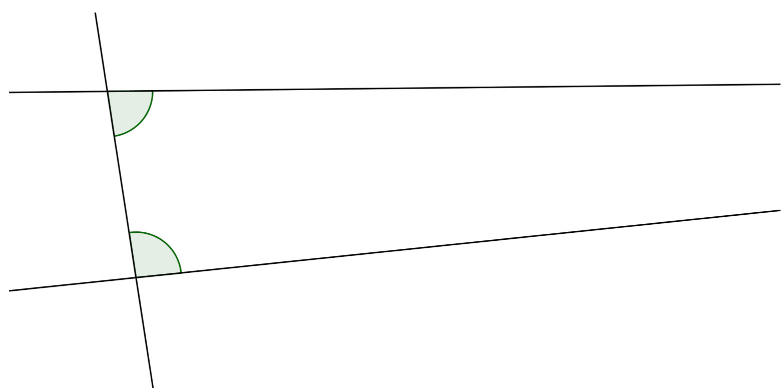
Exercise 8 Suppose that Axiom 2 in Young's Geometry were changed to read that "there are exactly two points on every line." How many points and lines would the geometry have? What if every line had four points? Can you generalize your result? Suppose that "there are exactly n points every line." How many points and lines would the geometery have? †

5 Searching for Euclid

Euclid's *Elements* introduced both axiomatics and formal geometry to the world. Though many of the results were probably known before Euclid, his book survived from antiquity and for more than 2000 years served as the main textbook for students of geometry. The 19th century produced a revolution in formal logic and the study of axiomatics. One result was that it became clear that Euclid's system was both incomplete and, insofar as it went, failed to meet the modern standards of rigor. The great German mathematician David Hilbert attempted to rectify these issues and published a new system of axioms in his book *Grundlagen der Geometrie* in 1898.

Hilbert's set of axioms was independent and complete. His formulation also split the axioms into differents sets: **incidence** axioms, **betweeness** axioms and **congruence** axioms. His idea is obvious: he knew the properties and theorems needed to make Euclidean Geometry work "correctly," and he developed structures within each set of axioms to force those properties into the system. Another important set of independent and complete Euclidean axioms was provided by Birkhoff. Where Hilbert had more than a dozen axioms, Birkhoff's elegant approach had only four axioms! The final important set of Eculidean axioms we will work with is a dependent system but was designed for use in secondary mathematics textbooks (SMSG). More axioms are given than are needed, but students aren't stuck proving theorems like "lines are straight." The additional axioms move the starting point into interesting and important ideas for school mathematics.

The study of Euclidean Geometry has to focus on the Fifth Postulate (Parallel Postulate) which has been controversial since antiquity. It appears even Euclid worried about it. Greek geometers argued it wasn't an axiom, it was a theorem. The idea is simple: "Two parallel lines never meet," and its statement refers to the following figure:



Here is Euclid's Fifth Postulate in full: "If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles." Attempts to prove it from the other axioms failed. Attempts to explain why included the idea that Euclid hadn't stated his postulate properly. The various restatements and proof attempts actually led to many important discoveries and, eventually, to non-Euclidean geometry.

We must use a properly restated (and equivalent) version of it to create Euclidean geometry. The most famous is the Playfair Postulate: "In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point." This is the modern idea of parallel. The "at most" turns out to be enough because, using the other Euclidean axioms, we can prove there is exactly one.
The study of non-Euclidean geometry negates the Playfair Postulate in one of two ways:

- In a plane, given a line and a point not on it, **more than one line** parallel to the given line can be drawn through the point.

- In a plane, given a line and a point not on it, **no line** parallel to the given line can be drawn through the point.

On our way to Euclidean Geometry, we can ask the following question: "What theorems can be proven without using any parallel axiom at all?" A set of Eudlidean axioms without a parallel axiom is called a **neutral geometry**.

Now our way forward is clear: we will start with Hilbert's version of Euclidean Geometry, thinking about each group of axioms. We will postpone any parallel postulate as long as possible and establish key results in neutral geometry. These results will be true regardless of the choice of parallel postulate we use. We will then use Playfair's Postulate (or a reasonable atlernative) to investigate those results which are only true in Euclidean Geometry before moving on to study non-Euclidean Geometry. The course notes for "Geometry: Axiom Sets" sections give four different formulations of Euclidean geometry, starting with Euclid himself. The axiomatic systems of Hilbert and Birkhoff follow. Both are rigorous in the modern sense and, as such, are complete, independent and have as a model the Euclidean plane. Finally, we have the SMSG axiom set which is rigorous, suitable for high school students, but which happens to be dependent.

6 Incidence Geometries

Hilbert's formulation of Euclid is very helpful. His first set of axioms describe **incidence** or connnection. We need to know when a line contains a point, or when two lines intersect at a given point. Incidence geometries can be either finite or infinite. These axioms can apply to our finite geometries but will also be flexible enough for our purposes when we turn our attention to Euclidean and non-Euclidean geometries. Any geometry that satisfies the following three axioms is called an **Incidence Geometry**.

6.1 Incidence Axioms

Axiom I.1 For every pair of distinct points A and B there is a unique line l containing A and B.

Axiom I.2 Every line contains at least two points.

Axiom I.3 There are at least three points that do not lie on the same line.

Note that we have extended the undefined term "on" to include the terms"lie on" and "contains."

Theorem 1 If two distinct lines intersect, then the intersection is exactly one point.

Theorem 2 For each point there exist at least two lines containing it.

Theorem 3 There exist three lines that do not share a common point.

Theorem 4 Fano's Geometry is an incidence geometry.

Theorem 5 Young's Geometry is an incidence geometry.

7 Parallel Postulates

If you think carefully, you realize that the idea of "parallel" is closely connected to the ideas of incidence. Consider the following situation while keeping in mind the Incidence Axioms:

Parallel Possibilities. Given an arbitrary line l and any point P not on l, three possibilities exist:

(i.) There exist **no lines** on P parallel to l.

(ii.) There exists **exactly one line** on P parallel to l.

(iii.) There exists **more than one line** on P parallel to l.

Option (ii) leads to Euclidean geometry, and any geometry whose axioms imply some equivalent statement is said to have the Euclidean parallel property. If we chose either of the other options, we will have a non-Euclidean geometry.

Theorem 6 The four-point geometry has the Euclidean paralllel property.

Theorem 7 Prove or disprove. Fano's geometry has the Euclidean paralllel property because one of its axioms is equivalent to alternative 2 above.

Theorem 8 Prove or disprove. Young's geometry has the Euclidean paralllel property because one of its axioms is equivalent to alternative 2 above.

7.1 Models for Incidence Geometries

Each model below is an interpretation of the undefined terms **points** and **lines**. For each one, (a) determine whether it is an incidence geometry and (b) which of the three parallel postulate alternative it would satisfy.

Model 1 Points are points on a Euclidean plane, and lines are non-degenerate circles in the Euclidean plane.

Model 2 Points are points on a Euclidean plane, and lines are all those lines on the plane that pass through a given fixed point P.

Model 3 Points are points on a Euclidean plane, and lines are concentric circles all having the same fixed center.

Model 4 Points are Euclidean points in the interior of a fixed circle, and lines are the parts of Euclidean lines that interesect the interior of the circle.

Model 5 Points are points on the surface of a Euclidean sphere, and lines are great circles on the surface of that sphere.

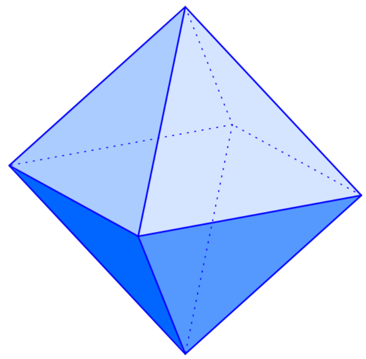
Model 6 Points are points on a Eucliden hemisphere (not including those points on the great circle that defines the hemisphere), and lines are great semicircles (e.g. those points on a great circle that intersect with the hemisphere).

Model 7 In Euclidean 3-space, points are interpreted as lines and lines are interpreted as Euclidean planes.

Model 8 Consider a finite geometry where the points are interpreted to be the vertices of a regular octahedron, and lines are sets of exactly two points. How many lines does it have? (A picture of a regular octahedron – not necessarily the exact model for this geometry – is shown below.)

Definitions for Incidence Geometries

- **Affine.** An incidence geometry that exhibits the Euclidean parallel property is called an **affine geometry**.
- **Projective.** An incidence geometry that has no parallel lines and in which each line has at least three points is called a **projective geometry**.
- **Parallel.** Two lines that do not intersect are called **parallel** lines.
- **Concurrent.** Three or more lines that intersect on the same point are called **concurrent**.



Theorem 9 In an affine geometry all lines must contain the same number of points.

Theorem 10 In an arbitrary affine geometry, consider intersecting lines m and n. If line l is parallel to either m or n, it intersects the other.

Theorem 11 In an arbitrary **finite** affine geometry, if every line contains exactly n points, then every point has exactly $n + 1$ lines on it.

Theorem 12 In an arbitrary **finite** affine geometry, if every line contains exactly n points, then there are exactly n^2 points and $n(n + 1)$ lines in the geometry. †

Theorem 13 Fano's geometry is projective.

Exercise 9 In Fano's geometry, axiom 4 is replaced by "There are exactly four points on every line." Devise a concrete model for this geometry. †

Exercise 10 Conjecture whether the alteration of Fano's geometry in Exercise 9 is affine. Prove your conjecture.

Exercise 11 Conjecture whether the alteration of Fano's geometry in Exercise 9 is preojective. Prove your conjecture.

8 Euclid

We have discussed several limitations of Euclid's **Elements**. The three key flaws are:

- Failure to recognize need for undefined terms.
- Use of subtle but unstated postulates in the proof of theorems.
- Reliance on diagrams to guide the logic of proofs.

We will discuss each of these in more depth during class.

8.1 Consequences of Euclid's Parallel Postulate

The following statements are all direct (although not neccessarily immediate) consequences of Euclid's 5th Postulate:

1. Through any given point, only one line can be drawn that is parallel to a given line.
2. There exists at least one triangle in which the sum of the measures of the interior angles is 180°.
3. Parallel lines are everywhere equidistant.
4. There exist two triangles that are simlar but not congruent.
5. There exist two straight lines that are equidistant at three different points.
6. Every triangle can be circumscribed.
7. The sum of the mesaures of the interior angles of a triangle is the same for all triangles.
8. A rectangle exists.
9. Rectangles can be constructed using a compass and a straightedge.

The list is vital for two reasons. First, each statement is true if and only if Euclid's 5th Postulate is true. Second, if any one of them could be proven using Euclid's first four postulates, then Euclid's 5th Postulate is a theorem, not an axiom.

As a matter of fact, mathematicians from antiquity have attempted variations on this theme, and these efforts pushed the boundaries of our knowledge of Euclidean geometry forward – even when the mathematicians failed. The culmination of these investigations resulted in Playfair's Postulate: "In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point." This is the modern idea of **Euclid's Parallel Property**. The "at most" turns out to be enough because, using the other Euclidean axioms, we can prove there is exactly one.

8.2 Compass and Straightedge Constructions

One interesting aspect of Greek geometry was how they idolized compass and straightedge constructions. Not only did they perform constructions as examples, they used them in proofs in two ways. They illustrated proofs with constructions, but they also used the constructions themselves as proofs. If you think about it, these constructions are based on a set of axioms. A compass draws a circle with a given radius, so we have a way to establish congruence of line segments. A straightedge provides for "straightness," so that things like parallel and perpendicular have concrete meaning. And that's it. Every construction proof relies only on these ideas. Therefore, the steps we take during the contruction illustrate a formal proof of the idea.

No geometry course would be complete without multiple constructions demonstrated and proved. The first step is simply to demonstrate the construction which, for our class, will count toward's students' Moore Method proof points. The second step is to prove formally that the required relationships exist. For class, this will count a different proof attempt for Moore Method proof points. For presentation purposes, we will make our constructions in Geogebra. However, if you wish to have constructions in your notes, you should buy a **good** compass and a straightedge and bring them class. While the traditional compasses can be high quality, the point requires careful use and can rip your paper. Here is an inexpensive alternative:

They're available for about $5.

One interesting aspect of compass and straigtedge constructions is the set that are simply impossible. The proof of the following theorem requires some rather elegant uses of Abstract Algebra which wasn't invented until the 18th century. However, we will take the following as true, without offering a proof:

Impossible Constructions. These three classical constructions are impossible using only a compass and straightedge:

IC.1 Trisecting an angle

IC.2 Squaring a circle - constructing a square with the same area as a given circle

IC.3 Doubling a cube - constructing a cube with volume exactly twice that of a given cube

The interesting thing is that, even today, folks refuse to believe they're impossible and continue to offer "proofs" of one or another. The scientific word for this line of inquiry is "pseudomathematics," and its practitioners are called "cranks." You can check out Rationalwiki's Psuedomathematics page and can find many other hilarious stories of mathematical inanity if you use google and the search query "math cranks." Have fun. But trust me, the proof that these three constructions are impossible is ironclad. No credible mathematician doubts it. Not in the least.

Here is a list of constructions that are **possible**, and that we will demonstrate and prove in class:

Lines

L.1 Copy a line segment

L.2 Sum of line segments

L.3 Difference of two line segments

L.4 Perpendicular bisector of a line segment

L.5 Divide a line segment into n equal segments

L.6 Perpendicular to a line at a point on the line

L.7 Perpendicular to a line from an external point

L.8 Perpendicular to a ray at its endpoint

L.9 A parallel to a line through a point (angle copy method)

L.10 A parallel to a line through a point (rhombus method)

L.11 A parallel to a line through a point (translated triangle method)

Angles

A.1 Copy an angle

A.2 Bisect an angle

A.3 Construct a 30° angle

A.4 Construct a 45° angle

A.5 Construct a 60° angle

A.6 Construct a 90° angle (right angle)

A.7 Sum of n angles

A.8 Difference of two angles

A.9 Suppplementary angle

A.10 Complementary angle

A.11 Constructing 75°, 105°, 120°, 135°, 150° angles and more

Triangles

T.1 Copy a triangle

T.2 Triangle, given all 3 sides (SSS)

T.3 Triangle, given one side and adjacent angles (ASA)

T.4 Triangle, given two sides and included angle (SAS)

T.5 Triangle, given two sides and non-included angle (AAS)

T.6 Isosceles Triangle, given base and one side

T.7 Isosceles Triangle, given base and altitude

T.8 Isosceles Triangle, given leg and apex angle

T.9 30-60-90 right triangle given the hypotenuse

T.10 Equilateral Triangle

T.11 Midsegment of a Triangle

T.12 Medians of a Triangle

T.13 Altitudes of a Triangle

T.14 Altitudes of a Triangle (outside case)

Right Triangles

T.15 Right Triangle, given hypotenuse and one leg (HL)

T.16 Right Triangle, given both legs (LL)

T.17 Right Triangle, given hypotenuse and one angle (HA)

T.18 Right Triangle, given one leg and one angle (LA)

Triangle Centers and Circles

T.19 Incenter of a Triangle

T.20 Circumcenter of a Triangle

T.21 Centroid of a triangle

T.22 Orthocenter of a Triangle

T.23 Incircle (inscribed circle) of a Triangle

T.24 Circumcircle (circumscribed circle) of a Triangle

Circles and Tangents

C.1 Constructing the center of a circle or arc

C.2 Finding the center of a circle or arc with any right-angled object

C.3 Tangents to a circle through an external point

C.4 Tangent to a circle through a point on the circle

C.5 Tangents to two circles (external)

C.6 Tangents to two circles (internal)

C.7 Circle through three points

Ellipses: Finding the foci of a given ellipse

Polygons

P.1 Square given one side

P.2 Square inscribed in a circle

P.3 Regular pentagon

P.4 Regular hexagon

P.5 Regular n-gon (some are not possible!)

You can find worksheets to help you begin (most of) the constructions on the above list at:

http://www.mathopenref.com/worksheetlist.html

Another interesting branch of constructions merges with the ideas of compass and straightedge. Origami (paper folding) is not just an art form, it can also be used in a geometry course for constructions. Consider how you would bisect and angle with paper-foling. Some "impossible constructions" in the compass and straigthedge world are possible with paper-folding. For example, you can trisect and angle with paper-folding. Many of the compass and straightedge constructions can also be performed with paper-folding or with paper-folding plus compass and straightedge. For use in the K12 schools, all of these paper-folding constructions are great learning tools.